TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 199, 1974

SMOOTH COMPLEX PROJECTIVE SPACE BUNDLES AND $B\widetilde{U}(n)$

BY

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ABSTRACT. Smooth fiberings with complex projective and Dold manifold fibers are studied and a bordism classification for even complex projective space bundles is given. The Z_2-cohomology of $B\widetilde{U}(n)$ is computed with its Steenrod algebra action.

1. **Introduction.** Let H^* be a Z_2-Poincaré algebra [3], d a formal class of degree 2 and b in H^1. In [2] it was shown that if $H^*[d]$ is given the Steenrod algebra structure determined by $Sq^1d = bd$ and if $\Sigma_{i=0}^n(1 + b)^{n-i}a_i$ is an "sw-class" in H^*, where a_i is in H^{2i}, then

$$K^* = H^*[d]/\langle d^n + a_i d^{n-1} + \cdots + a_n\rangle$$

is a Poincaré algebra. It was also shown that if K^*, as above, is a Poincaré algebra where d is in K^2, H^* is a Poincaré algebra, a_i are in H^{2i} and $Sq^1d = bd$ for some b in H^1, then $\Sigma_{i=0}^n(1 + b)^{n-i}a_i$ is an sw-class in H^*.

In this paper, we will use the above result to characterize those unoriented bordism classes which have a representative which fibers smoothly (over another manifold) with fiber an "even" complex projective space, $CP(2k)$. See [6] for the case $k = 1$ as well as for fiberings with real projective fibers. We discuss $P(n, m)$ fiberings (where $P(n, m)$ denotes the Dold manifold $S^n \times_{Z_2} CP(m)$) and show that "most" (unoriented) bordism classes contain a representative which fibers with $P(1, 2)$ as fiber. To get our results, we need to consider $B\widetilde{U}(n)$, the classifying space of $\widetilde{U}(n)$, see [5], which is to sw-pairs as $BO(n)$ is to sw-classes.

All algebras will be over Z_2 and cohomology will be singular theory with Z_2 coefficients. If η is a bundle, then $E(\eta)$ and $B(\eta)$ denote the total and base spaces of η. $RP(\eta)$ will denote the real projective space bundle associated with η and $CP(\eta)$ the complex projective space bundle (if η is complex). Γ and Λ will denote the canonical line bundles (real and complex, respectively)

Received by the editors October 15, 1973.

AMS (MOS) subject classifications (1970). Primary 55F10, 57D75; Secondary 55F40, 57D20.

Key words and phrases. Smooth fibrations, classifying spaces, Dold manifolds, bordism, Stiefel-Whitney class.

over $RP(\eta)$ and $CP(\eta)$. γ_n and λ_n will denote the universal bundles over $BO(n)$ and $BU(n)$ and for $n = 1$, the "universal" line bundles over $RP(m)$ and $CP(m)$.

The author wishes to thank his major advisor, Professor R. E. Stong of the University of Virginia, for his encouragement and help during the preparation of the author's thesis, of which this paper is a part.

2. $B\widetilde{U}(n)$. Let $\widetilde{U}(n)$ be the subgroup of $O(2n)$ generated by $U(n)$ and conjugation. The inclusion of $\widetilde{U}(n)$ in $O(2n)$ gives a map on classifying spaces, which we will call j. There is the homomorphism from $\widetilde{U}(n)$ to Z_2 given by dividing out $U(n)$ which yields a fibration of $B\widetilde{U}(n)$ over BZ_2 with fiber $BU(n)$. Call the projection π and the fiber inclusion i. According to Stong [5],

$$H^*(B\widetilde{U}(n)) \cong Z_2[\pi^*(\iota), j^*(w_2), \cdots, j^*(w_{2n})],$$

where ι is nonzero in $H^1(BZ_2)$ and $w_{2k} = w_{2k}(\gamma_{2n})$. Denote $j^*(\gamma_{2n})$ by η.

Let δ be the $CP(n-1)$ bundle associated with the universal principal $\widetilde{U}(n)$ bundle over $B\widetilde{U}(n)$ and $E(\hat{\delta})$ be the total space of the associated $E(\lambda_1)$ bundle over $B\widetilde{U}(n)$. Then $\hat{\delta}$ is a fibering of $E(\hat{\delta})$ over $E(\delta)$, which pulls back, via the inclusion of $CP(n-1)$ in $E(\delta)$, to λ. Hence

$$H^*(E(\delta)) \cong H^*(B\widetilde{U}(n))[d]/\langle d^n + \alpha_1 d^{n-1} + \cdots + \alpha_n\rangle,$$

where $d = w_2(\hat{\delta})$ and α_i is in $H^{2i}(B\widetilde{U}(n))$. (To see α_i in terms of $w_{2i}(\eta)$, see [1].)

Similarly, if ψ denotes the $RP(2n-1)$ bundle associated with the universal principal $\widetilde{U}(n)$ bundle over $B\widetilde{U}(n)$ and $E(\hat{\psi})$ denotes the corresponding $E(\gamma_1)$ bundle, then there is a fibering $\hat{\psi}$ of $E(\hat{\psi})$ over $E(\psi)$ which pulls back to γ_1 over the fiber of ψ. Moreover ψ classifies naturally into $RP(\gamma_{2n})$, which gives that

$$H^*(E(\psi)) \cong H^*(B\widetilde{U}(n))[c]/\langle c^{2n} + w_1(\eta)c^{2n-1} + \cdots + w_{2n}(\eta)\rangle;$$

where $c = w_1(\hat{\psi})$.

LEMMA 2.1. *$H^*(E(\psi)) \cong H^*(E(\delta))[c]/\langle c^2 + \beta' C + d\rangle$, where $\beta' = w_1(\hat{\delta})$.*

PROOF. The sphere of $\hat{\delta}$ is the S^{2n-1} bundle associated with the universal principal $\widetilde{U}(n)$ bundle and hence $RP(\hat{\delta})$ is $E(\psi)$ which fibers over $E(\delta)$ with $RP(1)$ as fiber. Note that $\hat{\psi}$ pulls back to γ_1 over this fiber. The asserted relation is the usual one for the projective bundle associated with a vector bundle. □

Since β' is in $H^1(E(\delta))$, $\beta' = \pi_\delta^*(\beta)$ for some β in $H^1(B\widetilde{U}(n))$.

LEMMA 2.2. $\beta = \pi^*(\iota)$

PROOF. It suffices to show that $\beta \neq 0$. Let $\tilde{U}(n)$ act on $S^m \times U(n)$ via the antipodal map and conjugation, which is a principal action. Classifying it and pulling back δ, we get the usual fibration of $P(m, n-1)$, the Dold manifold, over $RP(m)$ and a map of this bundle into δ.

It is known that (see [8]):

(i) $H^*(P(m, n-1)) \cong Z_2[c, d]/\langle c^{m+1}, d^n \rangle$, where the degree of c is one and that of d is two.

(ii) The Stiefel-Whitney class of $S^m \times_{Z_2} E(\lambda_1) = S^m \times U(n) \times_{\tilde{U}(n)} E(\lambda_1)$, as a two plane bundle over $P(m, n-1)$, is $1 + c + d$.

Therefore β' pulls back to c in $H^1(P(m, n-1))$ and is nonzero. □

If $x = 1 + x_1 + x_2 + \cdots + x_n$, where x_i is a $2i$-dimensional class and y is one dimensional, we will call (x, y) an "sw-pair" if $\Sigma_{j=0}^{n} x_j(1 + y)^{n-j}$ is an sw-class.

THEOREM 2.3. (i) $H^*(B\tilde{U}(n)) \cong Z_2[\beta, \alpha, \cdots, \alpha_n]$;

(ii) (α, β) *is an sw-pair, where* $\alpha = 1 + \alpha_1 + \cdots + \alpha_n$;

(iii) *If* (a, b) *is an sw-pair in a left unstable* $A(2)$ *algebra* X^*, *then there is an* $A(2)$-*homomorphism* σ: $H^*(B\tilde{U}(n)) \to X^*$ *with* $\sigma(\alpha) = a$ *and* $\sigma(\beta) = b$.

PROOF. On $H^*(E(\psi))$, there are the relations $\Sigma_{j=0}^{2n} c^{2n-j} w_j(\eta) = 0$, $\Sigma_{k=0}^{n} d^{n-k}\alpha_k = 0$ and $d = c^2 + c\beta$. Hence

$$\sum_{i=0}^{2n} c^i w_{2n-i} = \sum_{j=0}^{n} \alpha_j (c^2 + \beta c)^{n-j},$$

identically in c. Hence $w(\eta) = \Sigma\, \alpha_j(1 + \beta)^{n-j}$, which gives part (ii). Moreover,

$$w_{2k} = \sum_{j=0}^{k} \binom{n-k+j}{2j} \alpha_{k-j}\beta^{2j}.$$

Therefore w_{2k} and α_k are equally acceptable polynomial generators and Stong's result cited above gives (i).

To finish, it is enough to show that the epimorphism from the cohomology of the product

$$K(Z_2, 2) \times K(Z_2, 4) \times \cdots \times K(Z_2, 2n) \times K(Z_2, 1)$$

of Eilenberg-Mac Lane spaces to $H^*(B\tilde{U}(n))$, defined by α and β, has for kernel precisely those relations imposed by (α, β) being an sw-pair. There are unique polynomials $p_{ij}(x, y_1, \cdots, y_n)$ with $Sq^i\alpha_j + p_{ij}(\beta, \alpha_1, \cdots, \alpha_n) = 0$ for all

i and j. Suppose the cohomology of the above product is generated (as an $A(2)$ algebra) by $\iota_1, \iota_2, \cdots, \iota_{2n}$. Let the ideal J be generated by the elements $Sq^i\iota_{2j} + p_{ij}(\iota_1, \cdots, \iota_{2n})$. Let $K^* = Z_2[\iota_1, \cdots, \iota_{2n}]$. Let L^* be the cohomology of the above product and denote the epimorphism to $H^*(B\widetilde{U}(n))$ by τ. Note that τ restricted to K^* is a ring isomorphism and that the projection of L^* to L^*/J is an epimorphism when restricted to K^*. Since J is in the kernel of τ, τ factors through L^*/J and must give an isomorphism between L^*/J and $H^*(B\widetilde{U}(n))$. □

3. **$CP(2k)$-fibrations and bordism.** We wish now to connect Theorem 2.3 with the result cited in the introduction. Our main result is:

THEOREM 3.1. *The ideal in N_*, the unoriented bordism ring, of classes having representatives which fiber over closed smooth manifolds with fiber $CP(2k)$ is the image of $N_*(B\widetilde{U}(2k+1))$ in N_* of the homomorphism which sends the class of the $\widetilde{U}(2k+1)$ bundle over M to the class of the total space of its associated $CP(2k)$ bundle.*

(Compare, in [6], Proposition 8.5 and the remarks following 8.6.)

PROOF. Suppose $\pi: M \longrightarrow P$ is a smooth fibration of closed manifolds with $CP(2k)$ as fiber. Since $w_2(M)$ must pull back nontrivially to the generator of the fiber, $H^*(M)$ is freely generated, as an $H^*(P)$ module, by classes $1, e, \cdots, e^{2k}$, where $e = w_2(M)$. Moreover, there will be a relation $\Sigma_i\, e^i f_{2k+1-i} = 0$ which will give the product, where f_j is in $H^j(P)$.

If $Sq^1 e = be + g$, then $Sq^1 f_1 = bf_1 + g$ (applying Sq^1 to the above relation). Setting $d = e + f_1$ and defining the class $a = 1 + a_1 + \cdots + a_{2k+1}$ by the relation

$$\sum_{i=0}^{2k+1} e^i f_{2k+1-i} = \sum_{i=0}^{2k+1} d^i a_{2k+1-i},$$

we conclude that (a, b) must be an sw-pair (see introduction).

Hence there is a homomorphism $\sigma: H^*(B\widetilde{U}(2k+1)) \longrightarrow H^*(P)$ taking β to b and α to a. The results of [3] imply that there is a manifold Q and a map $f: Q \longrightarrow B\widetilde{U}(2k+1)$ such that f^* and σ are bordant in the algebraic bordism of $H^*(B\widetilde{U}(2k+1))$. Let $M' = f^*(E(\delta))$. We claim that M' is bordant to M.

LEMMA 3.2. *The correspondence*

$$(H^*, (a, b)) \longrightarrow H^*[d]/\langle d^n + a_1 d^{n-1} + \cdots + a_n\rangle,$$

where (a, b) is an sw-class, H^* is a Poincaré algebra and d is a formal two-

dimensional class, defines a homomorphism from the mth algebraic bordism group of $H^*(B\widetilde{U}(n))$ *to* N_{m+2n-2}.

PROOF. Suppose $(H^*, (a, b))$ bounds. Then there is a self-annihilating, homogeneous subalgebra J^* in H^* which is closed under the left and right $A(2) \otimes H^*(B\widetilde{U}(n))$ action. J^* is the image of the bounding Lefschetz algebra. See [7].

Let $R^* = J^*[d]/\langle d^n + a_1 d^{n-1} + \cdots + a_n\rangle$ which is a homogeneous subalgebra of $K^* = H^*[d]/\langle d^n + \cdots + a_n\rangle$ and is closed under the left $A(2)$ action. One shows, by straightforward arguments, that R^* is self-annihilating and closed under the right $A(2)$ action. Hence K^* bounds. Since the correspondence is clearly additive, the result follows. □

The theorem now follows by the equivalence of N_* with the algebraic bordism of $H^*(pt)$. □

4. $N_*(B\widetilde{U}(n))$ **and** $P(n, m)$ **fibrations.** In this section, we find generators for $N_*(B\widetilde{U}(n))$ and the indecomposables in the image of $N_*(B\widetilde{U}(n)) \to N_*$, the homomorphism of the previous section. We also collect several related results on $P(n, m)$ fibrations.

There is an involution of $\widetilde{U}(n)$ (which on the included $U(n)$ is conjugation) whose fixed subgroup is $Z_2 \times O(n)$. The composition

$$\theta\colon Z_2 \times (O(1) \times \cdots \times O(1)) \to Z_2 \times O(n) \to \widetilde{U}(n)$$

is clearly the inclusion of a maximal torus and the induced homorphism

$$\theta^*\colon H^*(B\widetilde{U}(n)) \to H^*(BZ_2) \otimes \left\{\bigotimes_{i=1}^{n} H^*(BO(1))\right.$$

is a monomorphism.

LEMMA 4.1. $\theta^*(\beta) = y$ *and* $\theta^*(\alpha_j) = \sigma_j(x_1(y + x_1), \cdots, x_n(y + x_n))$, *where* σ_j *denotes the jth elementary symmetric function,* y *generates the cohomology of* BZ_2 *and* x_i *generates the cohomology of the ith factor* $BO(1)$.

PROOF. Since the composition, $Z_2 \times O(n) \to \widetilde{U}(n) \to Z_2$, is projection on the first factor, $\theta^*(\beta) = y$. We claim that η, the bundle over $B\widetilde{U}(n)$ given by the inclusion of $\widetilde{U}(n)$ in $O(2n)$, pulls back over $BZ_2 \times BO(n)$ to $(\gamma_1 \hat{\otimes} \gamma_n) + (1 \hat{\otimes} \gamma_n)$, where $\hat{\otimes}$ denotes the exterior tensor product of vector bundles. Clearly, this will complete the proof.

If $f\colon BO(n) \to BU(n)$ denotes the usual complexification (induced by the inclusion of $O(n)$ in $U(n)$), then the inclusion of $BZ_2 \times BO(n)$ in $B\widetilde{U}(n)$

classifies $EZ_2 \times_{Z_2} f^*(\lambda_n)$, where λ_n is a Z_2-space via conjugation. This space is

$$\{(s, y, a, b) \in S^\infty \times BO(n) \times R^\infty \times R^\infty : a \in y, b \in y\}$$

modulo the relation $(s, y, a, b) \sim (-s, y, a, -b)$, where we are thinking of $BO(n)$ as n-planes in R^∞. This bundle is $(1 \hat{\otimes} \gamma_n) + (S^\infty \times_{Z_2} \gamma_n)$, where

$$S^\infty \times_{Z_2} \gamma_n = \{(s, a, b) \in S^\infty \times BO(n) \times R^\infty : b \in a\}/(s,a,b) \sim (-s,a,-b).$$

Moreover

$$\gamma_1 \hat{\otimes} \gamma_n = \{(x, t, u, v) \in BO(1) \times R^\infty \times BO(n) \times R^\infty : t \in x, v \in u\},$$

modulo $(x, rt, u, v) \sim (x, t, u, rv)$ for any r in R^1. The correspondence $(s, a, b) \longrightarrow ([s], s, a, b)$ induces the isomorphism. □

Let $M(q, j_1, \cdots, j_n)$ be the product manifold

$$RP(q) \times RP(2j_1) \times RP(2j_1 + 2j_2) \times \cdots \times RP(2j_1 + \cdots + 2j_n).$$

Then there is the map

$$M(q, j_1, \cdots, j_n) \longrightarrow BZ_2 \times \prod_{i=1}^{n} BO(1) \longrightarrow B\widetilde{U}(n),$$

which we will denote by $f_{q,j_1,\cdots,j_n}$. Ordering the $(n+1)$-tuples $(q, j_1, \cdots, j_n)$ lexiographically, one easily shows, using the previous lemma, that $(q, j_1, \cdots, j_n) < (p, k_1, \cdots, k_n)$ implies that

$$f^*_{q,j_1,\cdots,j_n}(\beta^p \alpha_1^{k_1} \cdots \alpha_n^{k_n}) = 0.$$

It follows that the classes $[M(q, j_1, \cdots, j_n), f_{q,j_1,\cdots,j_n}]$ are an N_* basis for $N_*(B\widetilde{U}(n))$.

Our main result is:

THEOREM 4.2. *The image of the class of* $[M(q, j_1, \cdots, j_n), f_{q,j_1,\cdots,j_n}]$ *is decomposable in* N_* *if and only if the term of degree* p *in the expansion of*

$$\frac{\Sigma_{i=1}^n \{(1 + y + x_i)^{p+2n-2} + (1 + x_i)^{p+2n-2}\}}{\Pi_{i=1}^n (1 + y + x_i)(1 + x_i)}$$

is zero, where p *is the dimension of* $M(q, j_1, \cdots, j_n)$.

To demonstrate this, we need a preliminary result.

LEMMA 4.3. *If* $f: M \to B\widetilde{U}(n)$ *is a map, then*

$$w(f^*(\delta)) = \pi^*(w(M))(1+b)^{-1}\left\{\sum_{i=0}^{n}(1+b+d)^{n-i}a_i\right\},$$

where $a_i = f^*(\alpha_i)$ *and* $1+b+d = f^*(w(\hat{\delta}))$.

PROOF. Since $f^*(\delta)$ fibers smoothly over M (with fiber $CP(n-1)$), $w(f^*(\delta)) = \pi^*(w(M))w(\theta)$, where θ is the bundle along the fibers of $f^*(\delta) \to M$. We claim that θ is the pull back of a "universal" bundle $\widetilde{\theta}$, over δ, such that, when pulled back over the diagram

$$\begin{array}{ccc} S^\infty \times_{Z_2} CP\left(\bigoplus_{i=1}^{n}(\gamma_1 \otimes C)\right) & \longrightarrow & E(\delta) \\ \downarrow & & \downarrow \delta \\ BO(1) \times \prod_{i=0}^{n} BO(1) \to BO(1) \times BO(n) & \to & B\widetilde{U}(n), \end{array}$$

satisfies the relation:

$$(*) \qquad \hat{\delta} \otimes_R \pi^*\left(\bigoplus_{i=1}^{n}\gamma_1\right) \cong \widetilde{\theta} \oplus \det\hat{\delta} \oplus 1.$$

If $(*)$ holds, then $w(\widetilde{\theta}) = (1+\beta)^{-1}\,\Sigma_{i=0}^{n}(1+\beta+d)^{n-i}\alpha_i$, since $BO(1) \times \Pi_{i=1}^{n} BO(1) \to B\widetilde{U}(n)$ is monic on cohomology.

To prove $(*)$, we work over the double covers of δ and $B\widetilde{U}(n)$ defined by β. Pulling back η and $\hat{\delta}$, we receive the complex bundles $\hat{\lambda}_n$ and $\hat{\Lambda}$, of complex dimension n and 1 respectively. It is then standard that $\Lambda \otimes_C \pi^*(\hat{\lambda}_n) \cong S^\infty \times (1 \oplus \theta)$, where $\theta = \{(x, y) \in S(\lambda_n) \times E(\lambda_n): x \perp y\}$ modulo the usual S^1 action (here $\perp$ is as complex vectors). Pulling back to $BO(1) \times \Pi_{i=1}^{n} BO(1)$ and dividing out the Z_2 action gives $(*)$. □

PROOF OF THEOREM 4.2. Let

$$f^*_{q,j_1,\dots,j_n}(\delta) = X, \quad RP(f^*_{q,j_1,\dots,j_n}(\hat{\delta})) = Y \text{ and } k_l = j_1 + \dots + j_l.$$

Then

$$w(X) = (1+y)^q \prod_{i=1}^{n}(1+x_i)^{2k_i+1}\{(1+y+d)^n + \dots + a_n\},$$

where y and x_i generate the cohomology of $RP(q)$ and $RP(2k_i)$ respectively. Moreover,

$$w(Y) = w(X)\{(1+c)^2 + (1+c)y + d\},$$

where $c = w_1(f^*_{q,j_1,\ldots,j_n}(\hat{\psi}))$. Therefore

$$w(Y) = (1+y)^{q+1} \prod_{i=1}^{n} (1+x_i)^{2k_i+1} \left\{ \sum_{j=0}^{n} a_{n-j}(1+y+d)^j \right\}.$$

Let m be the dimension of X, so that $m = p + 2(n-1)$, and denote the mth s-class of Y by $s_m(Y)$; we have

$$s_m(Y) = (q+1)y^m + \sum_{i=1}^{n} x_i^m + s_m\left\{ \sum_{j=0}^{n} a_{n-j}(1+y+d)^j \right\}.$$

Since $m > q$ and $m > 2k_i$,

$$s_m(Y) = s_m\left\{ \sum_{j=0}^{n} a_{n-j}(1+y+d)^j \right\}$$

$$= \sum_{i=1}^{n} \{(y + x_i + c)^m + (c + x_i)^m\} = \sum_{j=0}^{m} \binom{m}{j} c^{m-j} \left\{ \sum_{i=1}^{n} \{(y+x_i)^j + x_i^j\} \right\}.$$

Hence

$$s_m(Y) = \binom{m}{1} c^{m-1} s_1(w) + \binom{m}{2} c^{m-2} s_2(w) + \cdots + \binom{m}{2n-2} c^{2n-2} s_{m-2n+2}(w),$$

where $w = w(f^*_{q,j_1,\ldots,j_n}(\eta))$. Since $c^{2n-1+i} \equiv \bar{w}_i c^{2n-1}$ modulo lower degree terms in c, $cs_m(Y)$ evaluates on the fundamental class of Y as does the expression

$$c^{2n-1}\left\{ \sum_{j=1}^{m-(2n-2)} \bar{w}_{m-(2n-2)-j} s_j(w) \right\}.$$

But for any $x \in H^*(M; Z_2)$, $c^{2n-1}x$ evaluates on the fundamental class of Y as x does on the fundamental class of M. Since $\bar{w} = \Pi_{i=1}^{n}(1+y+x_i)^{-1}(1+x_i)^{-1}$, the result follows. □

We will finish this section with several related results on smooth fiberings with Dold manifolds as fibers.

LEMMA 4.4. *If* $\pi\colon X \to M$ *is a smooth fibration with* $i\colon P(m, n) \to X$ *the inclusion of a fiber, then* $\pi_1(M)$ *acts trivially on* $H^*(P(m, n))$ *if either* $m \neq 2$ *or* n *is even.*

PROOF. Let $\theta\colon [0, 1] \to M$ with $\theta(0) = \theta(1) = x$. Then there is a diagram

$$\begin{array}{ccc} P(m, n) \times [0, 1] & \xrightarrow{\ \tilde{\theta}\ } & X \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ x \times [0, 1] & \xrightarrow{\ \theta\ } & M \end{array}$$

giving $\theta\colon P(m, n) \to P(m, n)$, defined by $\theta(p) = \tilde{\theta}(p, 1)$. Hence $\theta^*\colon H^*(P(m, n)) \to H^*(P(m, n))$ is a ring automorphism and a homomorphism of $A(2)$ algebras.

Since $H^*(P(m, n) \cong Z_2[c, d]/\langle c^{m+1}, d^{n+1}\rangle$, where the degrees of c and d are 1 and 2 respectively, $\theta^*(c) = c$. If $\theta^*(d) = d + c^2$, then

$$\theta^*(cd) = \theta^*(Sq^1 d) = Sq^1\theta^*(d) = Sq^1(d + c^2) = cd.$$

But $\theta^*(cd) = \theta^*(c)\theta^*(d) = c(d + c^2)$. Hence m is not greater than 2. If $m = 1$, then clearly $\theta^*(d) = d$.

According to [4], $\theta^*(w_i) = w_i$, where w_i is the ith Stiefel-Whitney class of $P(m, n)$. If $n = 2k$, $w = (1 + c)^m(1 + c + d)^{2k+1}$ (see [8]), then

$$\begin{aligned} w &= \left\{1 + \binom{m}{1}c + \binom{m}{2}c^2 + \cdots\right\}(1 + c + d)(1 + c^2 + d^2)^k \\ &= \left\{1 + c\left(\binom{m}{1} + 1\right) + c^2\left(\binom{m}{2} + \binom{m}{1} + \binom{k}{1}\right) + d + \cdots\right\}. \end{aligned}$$

Hence,

$$\left(\binom{m}{2} + \binom{m}{1} + k\right)c^2 + \theta^*(d) = \left(\binom{m}{2} + \binom{m}{1} + k\right)c^2 + d,$$

and $\theta^*(d) = d$. □

LEMMA 4.5. *If* $\pi\colon X \to M$ *and* $i\colon P(m, n) \to X$ *are as in the previous lemma and* n *is even, then* π *is totally nonhomologous to zero.*

PROOF. Let $n = 2k$ and $a = \binom{m}{2} + \binom{m}{1} + k$. Then, as above, $w_2(P(m, 2k)) = ac^2 + d$. Since π is locally trivial, $i^*(w_2(X)) = w_2(P(m, 2k))$ and $ac^2 + d$ is in the image of i^*. Hence $ac^2 + d$ is in the kernel of every differential d_i of the (cohomology) spectral sequence of π.

Since $d_2(ac^2 + d) = 0$, $d_2(d) = 0$. But $Sq^1(ac^2 + d) = Sq^1 d = cd$, which is therefore in the image of i^*. Hence, $0 = d_2(cd) = dd_2(c) + cd_2(d) = dd_2(c)$. Since $d_2(c)$ is in $H^*(M)$, $d_2(c) = 0$. But $d_3(c^2) = 0$ also and hence, $0 = d_3(ac^2 + d) = d_3(d)$.

Therefore, the spectral sequence is trivial and the result follows. □

THEOREM 4.6. *If* i: $P(1, 2k) \to X$ *is the inclusion of a fiber in the smooth fibration* π: $X \to M$, *X and M closed, then X is bordant to a manifold which fibers smoothly with fiber* $CP(2k)$.

PROOF. By Lemma 4.5, $H^*(X) \cong H^*(M)[1, c, d, cd, \cdots, d^i, cd^i, \cdots, cd^{2k}]$, as $H^*(M)$ modules. There are the relations $c^2 = \gamma c + \delta$ and $d^{2k+1} = \Sigma_{j=1}^{2k+1} a_j d^{2k-j+1}$, where γ is in $H^1(M)$, δ is in $H^2(M)$ and a_j is a $2j$-degree class in $K^* = H^*(M)[c]/\langle c^2 + \gamma c + \delta\rangle$.

Since $Sq^1\delta = \gamma\delta$, K^* is a Poincaré algebra (see [2]), $H^*(X)$ is a Poincaré algebra and $H^*(X) \cong K^*[d]/\langle d^{2k+1} + a_1 d^{2k} + \cdots + a_{2k+1}\rangle$. Since $2k$ is even, we can change generators, if necessary, to get $Sq^1 d = cd$. Hence the pair (a, c), where $a = 1 + a_1 + \cdots + a_{2k+1}$, is an sw-pair.

It follows that there is a homomorphism θ: $H^*(B\widetilde{U}(2k + 1)) \to K^*$ with $\theta(\beta) = c$ and $\theta(\alpha) = a$, and, as before, a pair (N, f) with f: $N \to B\widetilde{U}(2k + 1)$ bordant to (K^*, θ). Pulling δ back along f, we get a manifold fibering over N with fiber $CP(2k)$ which is bordant to X. (This uses Theorem 3.2.) □

THEOREM 4.7. *There are indecomposable manifolds which fiber over closed manifolds with fiber* $P(1, 2)$ *in all dimensions* m *of the form* $4k + 2$ *for* $k = 1, 2, \cdots$ *or* $2^p(2q + 1) - 1$ *for* $p > 0$ *and* $q > 0$ (*i.e., all odd dimensions, not of the form* $2^i - 1$).

PROOF. First note that for $q = 1$, the rational function of Theorem 4.2 becomes

$$my\left(\sum_{i=1}^{n} (1 + x_i)^{m-1}\right)\Big/\prod_{i=1}^{n} (1 + y + x_i)(1 + x_i).$$

Hence, we need the coefficient of $x_1^{2k_1} \cdots x_n^{2k_n}$ in

$$\frac{(1 + x_i)^{m-2}\{1 + (y + x_i) + (y + x_i)^2 + \cdots\}}{\Pi_{j=1;j\neq i}^{n}(1 + y + x_j)(1 + x_j)}.$$

Therefore, we want the coefficient of $x_1^{2k_1}\cdots x_n^{2k_n}$, deleting $x_i^{2k_i}$, in

$$\begin{aligned}&\{1+(x_1^2+y(1+x_1))+(x_1^2+y(1+x_1))^2+\cdots\}\\ &\qquad\qquad\cdots(1+x_n^2+\cdots)\cdots\\ &=(1+x_1^2+x_1^4+\cdots+y\psi_1)(1+x_2^2+x_2^4+\cdots+y\psi_2)\cdots.\end{aligned}$$

Therefore, the required coefficient is

$$m\left\{\binom{m-2}{2k_i}+\binom{m-2}{2k_i-1}+\cdots+\binom{m-2}{1}+\binom{m-2}{0}\right\}$$

which equals $m\binom{m-3}{2k_i}$. Hence, in this case,

$$s_m(X)=\sum_{i=1}^{n}\binom{m-3}{2k_i}=\sum_{i=1}^{n}\binom{k_1+\cdots+k_n+n-2}{k_i}.$$

Hence if $n=3$, $s_m(X)=\sum_{j=1}^{3}\binom{k_1+k_2+k_3+1}{k_j}$.

If $m=2^p(2q+1)-1$, $k_1+k_2+k_3=2^pq+2^{p-1}-2$. If $k_1+k_2+k_3=l+1$ is odd, then $\binom{l+1}{l}+\binom{l+1}{0}+\binom{l+1}{0}$ is odd and the manifold $S^1\times_{Z_2}CP((\gamma_1\otimes C)\hat{\oplus}1_C\hat{\oplus}1_C$ will do.

If $l+1$ is even, we claim that

$$\binom{l+1}{2^{p-1}-2}+\binom{l+1}{2^{p-1}q}+\binom{l+1}{2^{p-1}q-1}$$

is odd. To see this, set $q=\sum_{i=0}^{r}a_i2^i$ and note that

$$\binom{l+1}{2^{p-1}-2}\equiv\binom{a_r}{0}\cdots\binom{a_0}{0}\binom{0}{0}\binom{1}{1}\cdots\binom{1}{1}\quad\text{(modulo 2)},$$

which is odd. Moreover,

$$\begin{aligned}\binom{l+1}{2^{p-1}q}+\binom{l+1}{2^{p-1}q-1}&=\binom{l+2}{2^{p-1}q}=\binom{2^pq+2^{p-1}-1}{2^{p-1}q}\\ &=\binom{a_r}{0}\cdots\binom{a_i}{a_{i+1}}\cdots\binom{a_0}{a_1}\binom{0}{a_0}\binom{1}{0}\cdots\binom{1}{0},\end{aligned}$$

which is odd only if $a_0 = a_1 = \cdots = a_r = 0$. But $a_r = 1$. Hence $k_1 = 2^{p-1} - 2$, $k_2 = 2^{p-1}q$ and $k_3 = 2^{p-1}q - 1$ define a manifold which works if $m = 2^p(2q+1) - 1$ and $p > 1$.

To get the even dimensions, we want to consider the manifolds Q^n defined as follows. Let v be the smooth involution of $P(1, 2)$ defined by $v[(t_1, t_2), x] = [(-t_1, t_2), x]$, where t_i are in R^1, $t_1^2 + t_2^2 = 1$, x is in $CP(2)$ and $[\ ,\ ]$ denotes the usual equivalence class. Let $Q^n = S^n \times_{Z_2} P(1, 2)$, where the action takes (s, y) to $(-s, vy)$. Then

$$H^*(Q) \cong H^*(RP(n))[1, c, d, cd, d^2, cd^2],$$

as $H^*(RP(n))$ modules.

Now there is a diagram:

$$\begin{array}{ccc} CP(2) & \xrightarrow{i} & Q \\ \downarrow & & \downarrow{\scriptstyle p} \\ RP(1) & \xrightarrow{j} & S^n \times_{Z_2} RP(1) \\ & & \downarrow{\scriptstyle q} \\ & & RP(n) \end{array}$$

where p and q are fibrations with inclusions of fibers i and j respectively. Clearly $S^n \times_{Z_2} RP(1) = RP(\gamma_1 \oplus 1)$.

Letting T denote a tangent bundle and θ a bundle along the fibers, we have

$$T(Q) \oplus 2 \cong p^*(\Gamma \otimes q^*(\gamma_1 \oplus 1)) \oplus \theta_p \oplus p^*q^*((n+1)\gamma_1),$$

where Γ denotes the canonical line bundle over $RP(\gamma_1 \oplus 1)$. Let p' denote the usual fibration of $P(1, 2)$ over $RP(1)$, then $\theta_p = S^n \times_{Z_2} \theta_{p'}$. If σ' and ρ' denote the usual line and 2-plane bundle over $P(1, 2)$, $\sigma = S^n \times_{Z_2} \sigma'$ and $\rho = S^n \times_{Z_2} \rho'$, then $\theta_p \oplus \sigma \oplus 1 = 3\rho$. Let $c = w_1(\rho)$, $d = w_2(\rho)$ and x generate the cohomology of $RP(n)$. Then $c^2 = cx$ in $H^*(Q)$. Since Γ pulls back to σ via p, $w_1(\sigma) = c$. Hence $d^3 = 0$ in $H^*(Q)$.

It follows that

$$H^*(Q) \cong \frac{H^*(RP(n))[c]}{\langle c^2 + cx \rangle} \otimes \frac{Z_2[d]}{\langle d^3 \rangle},$$

as rings. Note that, since $Sq^1 d = cd$, the splitting is not as $A(2)$ algebras.

We have that

$$T(Q) \oplus 3 \oplus \sigma \cong p^*\{q^*((n+1)\gamma_1) \oplus (\Gamma \otimes q^*(\gamma_1 \oplus 1))\} \oplus 3\rho;$$

hence

$$\begin{aligned} w(Q) &= (1+x)^{n+1}(1+c+d)^3\{(1+c)^2 + x(1+c)\}(1+c)^{-1} \\ &= (1+x)^{n+1}(1+c+d)^3(1+c+x). \end{aligned}$$

It follows that

$$\begin{aligned} s_{n+5}(Q) &= \sum_{i=0}^{n+5} \binom{n+5}{i} c^i x^{n+5-i} + \sum_{p+2q=n+5} \binom{p+q-1}{q} c^p d^q \\ &= \binom{n+2}{2} c x^n d^2, \end{aligned}$$

since $c^i x^{n+5-i} = cx^{n+4} = 0$ for $i \geqslant 1$ and $c^p d^q = cx^{p-1} d^q = 0$, unless $q = 2$, $p - 1 = n$. But $\binom{n+2}{2}$ is odd precisely when $n \equiv 0$ or $1 \bmod 4$.

Hence Q^n is indecomposable in dimensions $4k+1$ and $4k+2$, which gives the result. □

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